Note on Stochastic Optimization

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This note is based upon reading of .

1 One-stage Stochastic Optimization

1.1 Various Modeling Way of Stochastic Optimization Problem

Consider a LP with uncertainty:

$$\begin{aligned}
&\min_x x_1 + x_2 \\
&\text{s.t.} \xi_1 x_1 + x_2 \geq 7 \\
&\quad \xi_2 x_1 + x_2 \geq 4 \\
&\quad x_1 \geq 0 \\
&\quad x_2 \geq 0.
\end{aligned} \tag{1}$$

If we solve this problem before realization of the uncertainty, then it is called *here and now model*; otherwise it is called *wait and see model*. There are various ways to solve a here and now model:

- Robust model (pessimistic): we consider the extreme case when $\xi = (1, 1/3)$ directly, under this case the feasible set is the smallest. The optimal solution is feasible for all possible realization but is too conservative.
- Robust model (optimistic): we consider the extreme case when $\xi = (4, 1)$ directly, under this case feasible set is the largest. However, the optimal solution is feasible with prob 0.
- Deterministic model (unbiased): we consider the mean value of ξ, i.e., $\xi = \mathbb{E}[\xi]$. The optimal solution is infeasible for some realization of ξ.
- Probabilistic model (chance constrained): Consider specific $\alpha_1, \alpha_2 \in (0, 1)$, we tend to solve

$$\begin{aligned}
&\min_x x_1 + x_2 \\
&\text{s.t.} P(\xi_1 x_1 + x_2 \geq 7) \geq \alpha_1 \\
&\quad P(\xi_2 x_1 + x_2 \geq 4) \geq \alpha_2 \\
&\quad x_1 \geq 0 \\
&\quad x_2 \geq 0.
\end{aligned} \tag{2}$$

- Recourse model (penalize shortfall): We consider the following optimization problem where the latter two parts are the expected shortfall of two resource constraints:

$$\min_x x_1 + x_2 + q_1 \mathbb{E}_{\xi_1}[(\xi_1 x_1 + x_2 - 7)^-] + q_2 \mathbb{E}_{\xi_2}[(\xi_2 x_1 + x_2 - 4)^-].$$

  This can also be reformulated as

$$\min_x x_1 + x_2 + \mathbb{E}[Q(x_1, x_2, \xi)],$$

  where Q is called recourse function and

$$\begin{aligned} \mathbb{E}[Q(x_1, x_2, \xi)] = \min_{y_1, y_2} \quad & q_1 y_1 + q_2 y_2 \\ \text{s.t.} \quad & \xi_1 x_1 + x_2 \geq 7 \\ & \xi_2 x_1 + x_2 \geq 4 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned} \tag{3}$$

1.2 General One-stage Stochastic Programming Models

2 Two-stage Stochastic Optimization

3 Multistage Stochastic Optimization

4 Sample Average Approximation

Definition 4.1 (Sample Average Approximation)

Consider a stochastic programming problem in the form

$$\min_{x \in X} \{f(x) := \mathbb{E}_\xi[F(x, \xi)]\},$$

where $X \subset \mathbb{R}^n$ is a given set and ξ is a random vector. Suppose ξ is discrete and has finite K realizations, denote its expected objective function as

$$f(x) = \mathbb{E}_\xi[F(x, \xi)]\} = \sum_{k=1}^{K} p_k F(x, \xi_k).$$

We generate a sample of N replications of ξ, note that $\xi^1, \cdots$ can be viewed as a sequence of random vectors with the same probability distribution as ξ. On the basis of these samples, we define a sample average function as

$$\hat{f}_N(x) := \frac{1}{N} \sum_{i=1}^{N} F(x, \xi^i).$$

Moreover, $\hat{f}_N(x)$ is an unbiased estimator of $f(x)$, i.e.,

$$\mathbb{E}_\xi[\hat{f}_N(x)]\} = f(x),$$

since each ξ^i has the same probability distribution as ξ.

Thus, the original problem can be approximated by

$$\min_{x \in X} \{\hat{f}_N(x)\}.$$

Note on *We consider SAA because of the computational difficulties in high dimension space.*